Math 111 Test 1 Version 1
Fall 2022

Solution.

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Please circle your lecture:

Santiago Arango 8:30	Maxwell Auerbach 2:30	Ariana Brown 4:00	Jasmine Camero 1:00	Alex Dunbar 8:30	Bryce Frederickson 1:00
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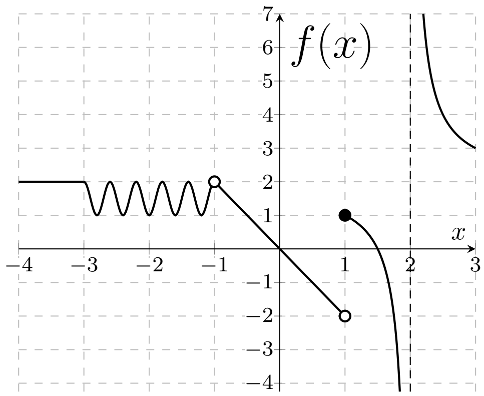
You have so many incredible strengths! Highlight one for yourself by writing it down below, before you begin this test.

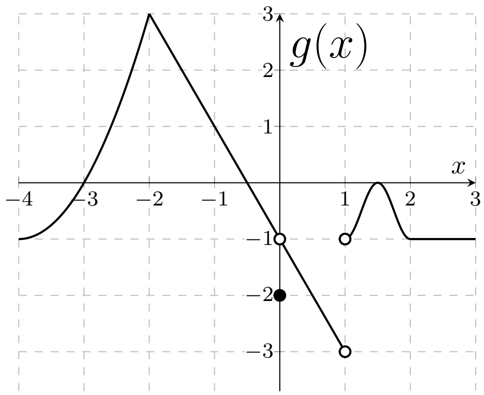
Directions—Please read carefully!

- You are not allowed to use a calculator or any other aids on this exam.
- The Emory Honor Code is in effect. No form of collaboration is permitted, whether with another student or with a non-student resource.
- Be sure to write neatly—illegible answers will receive little to no credit.
- You must show your work on Free-Response Questions in order to get full credit. An unsupported answer will get little to no credit.
- There is no partial credit on Multiple-Choice Questions.
- Do not detach pages or tamper with the exam packet. This test has XXXXX problems, and XXXXX pages.

Scratch paper. **Do not detach**. If you end up using this sheet for a solution, please indicate that on the problem.

1. (10 points) The graphs of functions f and g are given below.





Evaluate the following limits if they exist, or state "DNE". **If they do not exist give a short justification. Specifically, if a limit does not exist because it is ∞ or $-\infty$, please tell us whether it's ∞ or $-\infty$.**

(a) (2 points) $\lim_{x\to 1^+} g(x)$ $= -1$

(b) (2 points) $\lim_{x\to 0} g(x)$ $= -1$

(c) (3 points) $\lim_{x\to -3^+} \frac{f(x)}{g(x)}$ $= \infty$

$f(x)$ approaches 2 and $g(x)$ approaches 0.

(d) (3 points) $\lim_{x\to 2} \frac{g(x)}{f(x)}$ $= 0$

$g(x)$ approaches 1

$f(x)$ approaches $+\infty$ from the right and $-\infty$ from the left.

In any case $\frac{g(x)}{f(x)} \to 0$.

2. (12 points) Consider the function $f(t) = \dfrac{\sqrt{16+t}-4}{t}$. In this problem, we will determine $\lim\limits_{t\to 0} f(t)$.

(a) (2 points) Which of the following would be the most reasonable first step to compute $\lim\limits_{t\to 0} \dfrac{\sqrt{16+t}-4}{t}$? (*Select one*)

- ○ Once we plug in $t = 0$, we get $\frac{0}{0}$, hence the limit does not exist.
- ○ Once we plug in $t = 0$, we get 0, hence the limit is 0.
- ● We multiply both numerator and denominator by $\sqrt{16+t}+4$.
- ○ We factor $\sqrt{16+t}$ as $\sqrt{16}+\sqrt{t}$.

(b) (10 points) Evaluate $\lim\limits_{t\to 0} f(t)$, or state "DNE." If it does not exist give a short justification. Specifically, if a limit does not exist because it is ∞ or $-\infty$, please tell us whether it's ∞ or $-\infty$. **Show all your work.**

Note that for $t \neq 0$ we have:

$$\frac{\sqrt{16+t}-4}{t} = \frac{\sqrt{16+t}-4}{t} \cdot \frac{\sqrt{16+t}+4}{\sqrt{16+t}+4}$$

$$= \frac{\cancel{16}+t-\cancel{16}}{t(\sqrt{16+t}+4)}$$

$$= \frac{\cancel{t}}{\cancel{t}(\sqrt{16+t}+4)}$$

$$= \frac{1}{(\sqrt{16+t}+4)}.$$

Therefore,

$$\lim_{t\to 0} f(t) = \lim_{t\to 0} \frac{1}{(\sqrt{16+t}+4)} = \frac{1}{\sqrt{16}+4} = \frac{1}{8}.$$

3. (12 points) Consider the function $f(x) = \ln\left(\frac{1-x}{4-5x+x^2}\right)$. In this problem, we will determine $\lim_{x\to 3} f(x)$.

(a) (2 points) Which of the following would be the most reasonable first step to compute $\lim_{x\to 3} \ln\left(\frac{1-x}{4-5x+x^2}\right)$? (*Select one*)

Theese are incorrect.

- ○ Once we plug in $x = 3$, we get $\ln\left(\frac{0}{0}\right)$, hence the limit does not exist.
- ○ Once we plug in $x = 3$, we get $\ln(0) = -\infty$.
- ○ We clear denominators by multiplying top and bottom by $4 - 5x + x^2$.
- ● We factor the $4 - 5x + x^2$ in the denominator.

(b) (10 points) Evaluate $\lim_{x\to 3} f(x)$, or state "DNE." If it does not exist give a short justification. Specifically, if a limit does not exist because it is ∞ or $-\infty$, please tell us whether it's ∞ or $-\infty$. **Show all your work.**

We could just evaluate, since f is continuous at $x = 3$.

$$\lim_{x\to 3} f(x) = f(3) = \ln\left(\frac{1-3}{4-15+9}\right) = \ln\left(\frac{-2}{-2}\right) = \ln(1) = 0.$$

We could also follow the sugestion from part (a):

$x^2 - 5x + 4 = (x-1)(x-4)$, so that

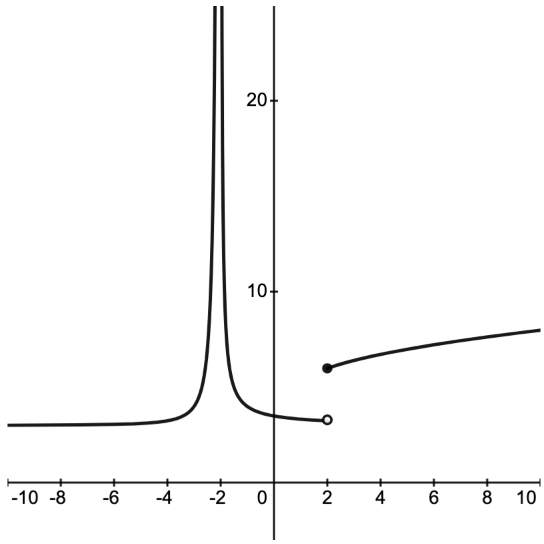
$$f(x) = \ln\left(\frac{-(x-1)}{(x-1)(x-4)}\right) = \ln\left(\frac{-1}{x-4}\right)$$

$$= \ln\left(\frac{1}{4-x}\right) = \cancel{\ln(1)}^{0} - \ln(4-x) = -\ln(4-x).$$

Thus:

$$\lim_{x\to 3} f(x) = -\ln(4-3) = -\ln(1) = 0.$$

4. (10 points) Below is the graph of a function f. Indicate on the graph the points at which f is discontinuous. Using limits, explain why f is discontinuous at each of those points. **Be sure to explain your reasoning. Answers with no justification will receive little to no credit.**



- f is discontinuous at $x = -2$ because it is not defined there. In particular,

$$\lim_{x \to -2} f(x) = \infty .$$

- f is discontinuous at $x = 2$ because the limit DNE. In particular,

$$\lim_{x \to 2^-} f(x) < \lim_{x \to 2^+} f(x) .$$

5. (16 points) Consider the function given by

$$f(x) = \frac{7 \cdot 5^x + 2^{2x} + 15}{5^x + 2^{2x} + 1}$$

(a) (2 points) Which of the following would be the most reasonable first step to compute $\lim_{x \to \infty} f(x)$? (*Select one*)

● We identify the dominant term, which is 5^x, and divide both numerator an denominator by it.

○ We identify the dominant term, which is 2^{2x}, and divide both numerator an denominator by it.

○ Once we plug in $x = \infty$, we get $\frac{\infty}{\infty}$, hence the limit does not exist.

○ Once we plug in $x = \infty$, we get $\frac{\infty}{\infty}$, hence the limit is equal to 1.

(b) Evaluate the following limits if they exist, or state "DNE". If they do not exist give a short justification. Specifically, if a limit does not exist because it is ∞ or $-\infty$, please tell us whether it's ∞ or $-\infty$. **Show all your work.**

i. (10 points) $\lim_{x \to \infty} f(x)$

The dominant term is 5^x because $2^{2x} = 4^x$, and $4^x < 5^x$ for $x > 0$. Dividing by 5^x in both numerator and denominator:

$$f(x) = \frac{7 + (4/5)^x + 15 \cdot 5^{-x}}{1 + (4/5)^x + 5^{-x}}$$

and

$\lim_{x \to \infty} f(x) = 7$, since $(4/5)^x, 5^{-x} \to 0$ when $x \to \infty$.

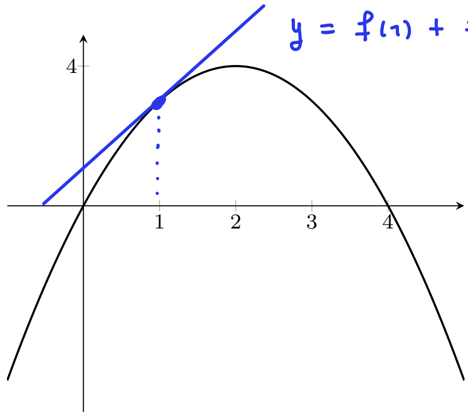
ii. (4 points) $\lim_{x \to -\infty} f(x)$

Since $4^x, 5^x \to 0$ when $x \to -\infty$

$\lim_{x \to -\infty} f(x) = 15.$

6. (15 points) Let $f(x) = 4x - x^2$. Below is a portion of its graph.

$y = f(1) + f'(1)(x-1)$.



(a) (3 points) On the graph above, sketch a line whose slope is $f'(1)$.

(b) (12 points) Use a limit definition of the derivative to find $f'(1)$. **No credit will be given if you use methods other than the limit definition of the derivative.**

Solution 1:

$$\lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{(4x - x^2) - (4 - 1)}{x - 1}$$

$$= \lim_{x \to 1} \frac{-(x^2 - 4x + 3)}{x - 1}$$

$$= -\lim_{x \to 1} \frac{(x-3)\cancel{(x-1)}}{\cancel{(x-1)}}$$

$$= -\lim_{x \to 1} (x-3) = 2 .$$

Solution 2:

$$\lim_{h \to 0} \frac{f(h+1) - f(1)}{h} = \lim_{h \to 0} \frac{4(h+1) - (h+1)^2 - 3}{h}$$

$$= \lim_{h \to 0} \frac{4h + \cancel{4} - h^2 - 2h - \cancel{1} - \cancel{3}}{h} = \lim_{h \to 0} \frac{\cancel{h}(2-h)}{\cancel{h}}$$

$$= \lim_{h \to 0} (2 - h) = 2 .$$

7. (15 points) Differentiate the following functions. **You do not need to simplify your answers.**

(a) (3 points) $f(x) = \pi^{26}$

$f'(x) = 0$.

(b) (3 points) $g(x) = \dfrac{4}{x^3}$

$g'(x) = \frac{d}{dx}\left(4x^{-3}\right) = 4\frac{d}{dx}\left(x^{-3}\right) = 4 \cdot \left(-3x^{-4}\right)$

$= \dfrac{-12}{x^4}$.

(c) (3 points) $s(x) = 7e^x + 2x$

$s'(x) = \frac{d}{dx}\left(7e^x\right) + \frac{d}{dx}(2x) = 7e^x + 2$.

(Problem **7** *continued*)

(d) (3 points) $f(x) = x^7(9x + e^x)$

$$f'(x) = \frac{d}{dx}\left[x^7 (9x + e^x) \right]$$

$$= \frac{d}{dx}(x^7) \cdot (9x + e^x) + x^7 \frac{d}{dx}(9x + e^x)$$

$$= 7x^6(9x + e^x) + x^7(9 + e^x) \,.$$

(e) (3 points) $g(x) = \dfrac{\sqrt{5x}}{x^2 + 1949}$

$$g'(x) = \frac{\frac{d}{dx}\left(\sqrt{5}\, x^{1/2}\right)(x^2 + 1949) - \sqrt{5x}\, \frac{d}{dx}(x^2 + 1949)}{(x^2 + 1949)^2}$$

$$= \frac{\frac{\sqrt{5}}{2\sqrt{x}}(x^2 + 1949) - \sqrt{5x}\,(2x)}{(x^2 + 1949)^2}$$

8. (10 points) Sara, the owner of a cupcake truck, has been experimenting with the price of the cupcakes she sells. Unsurprisingly, she has found that the number of cupcakes she sells per day depends on the price. Let $C(p)$ be the number of cupcakes she sells when the price of a cupcake is p cents.

So far, Sara has found that when the price of a cupcake is 300 cents, she sells 600 cupcakes a day.

(a) (3 points) What are the units of $C'(300)$?

$$\frac{\text{cupcakes per day}}{\text{cents}}$$

(b) (3 points) Do you expect $C'(300)$ to be positive or negative? Why?

One would expect the function $y = C(p)$ to be decreasing: when the price is higher we sell less cupcakes. Decreasing differentiable functions have negative derivative, so we expect:

$$c'(300) < 0.$$

(c) (4 points) Suppose $C'(300) = -5$. If Sara raises the price of cupcakes to 310 cents, approximately how many cupcakes do you expect her to sell per day?

We use linear approximation:

The approximating function is

$$L(p) = c(300) + c'(300)(p-300)$$

$$L(p) = 600 - 5\cdot(p-300).$$

Then:

$$c(310) \approx L(310) = 600 - 5\cdot(310-300)$$

$$= 600 - 50$$

$$= 550 \text{ cupcakes per day.}$$

Students – do not write on this page!

1. (10 points) ____________________
2. (12 points) ____________________
3. (12 points) ____________________
4. (10 points) ____________________
5. (16 points) ____________________
6. (15 points) ____________________
7. (15 points) ____________________
8. (10 points) ____________________

TOTAL (100 points) ______________