Continuity

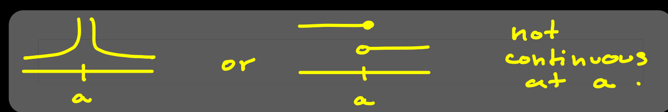
Def: A function f is continuous at a point a if

$$\lim_{x \to a} f(x) = f(a).$$

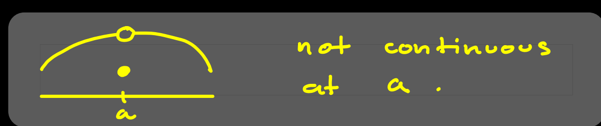
Note that for a function f to be continuous at a, we are implicitly requiring three things:

1) a is in the domain of f.

2) $\lim_{x \to a} f(x)$ exists.



3) $\lim_{x \to a} f(x) = f(a)$.



Example:

Consider the function

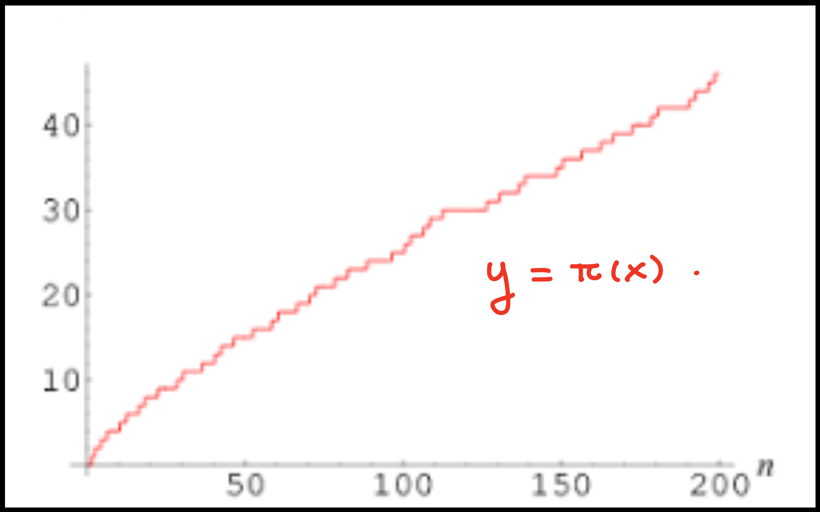
$\pi(x) :=$ number of primes less than or equal to x.

$\pi(1) = 0, \quad \pi(2) = 1, \quad \pi(3) = 2.$

$\pi(10) = 4$, because the primes ≤ 10 are $2, 3, 5, 7$.

Where is the function $\pi(x)$ discontinuous?

x	$\pi(x)$
10	4
10^2	25
10^3	168
10^4	1,229
10^5	9,592
10^6	78,498
10^7	664,579
10^8	5,761,455
10^9	50,847,534
10^{10}	455,052,511
10^{11}	4,118,054,813
10^{12}	37,607,912,018
10^{13}	346,065,536,839
10^{14}	3,204,941,750,802
10^{15}	29,844,570,422,669
10^{16}	279,238,341,033,925
10^{17}	2,623,557,157,654,233
10^{18}	24,739,954,287,740,860
10^{19}	234,057,667,276,344,607
10^{20}	2,220,819,602,560,918,840
10^{21}	21,127,269,486,018,731,928
10^{22}	201,467,286,689,315,906,290
10^{23}	1,925,320,391,606,803,968,923
10^{24}	18,435,599,767,349,200,867,866
10^{25}	176,846,309,399,143,769,411,680
10^{26}	1,699,246,750,872,437,141,327,603
10^{27}	16,352,460,426,841,680,446,427,399
10^{28}	157,589,269,275,973,410,412,739,598
10^{29}	1,520,698,109,714,272,166,094,258,063



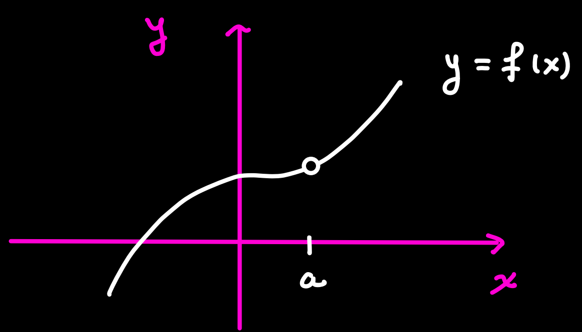
Types of discontinuity

A function f is discontinuous at a if it is not continuous at a, that is:

$$(*) \quad \lim_{x \to a} f(x) \neq f(a).$$

In this case, $(*)$ is called a discontinuity. There are several types of discontinuities, for example:

- **Removable discontinuity:**



When f is not defined at a but

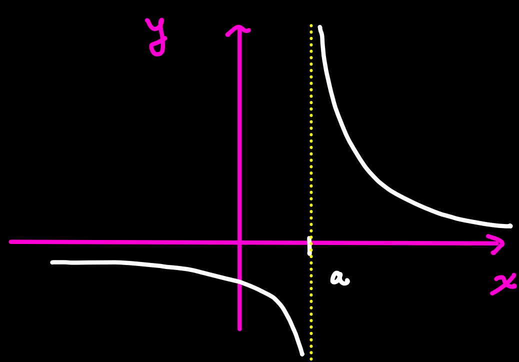
$$\lim_{x \to a} f(x) = L.$$

It is removable because we can extend f to a continuous function g by simply "filling the gap".

$$g(x) := \begin{cases} f(x), & \text{if } x \neq a, \\ L, & \text{if } x = a. \end{cases}$$

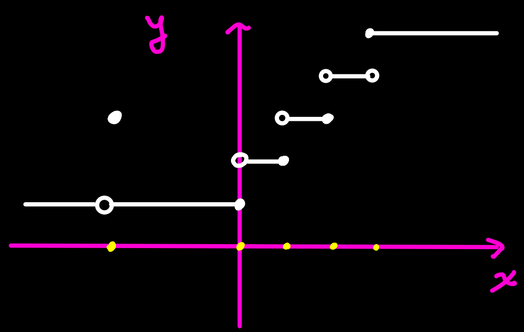
$\longrightarrow$ This function is continuous and $\lim_{x \to b} g(x) = \lim_{x \to b} f(x)$ for every b.

- **Infinite discontinuity:**



When f is not defined at a and has a vertical assymptote at the line $x = a$.

- **Jump discontinuity:**



a is a jump discontinuity when $f(a)$ exists but it is not equal to either $\lim_{x \to a^+} f(x)$ or $\lim_{x \to a^-} f(x)$.

Def: f is right-continuous at a if

$$\lim_{x \to a^+} f(x) = f(a).$$

f is left-continuous at a if

$$\lim_{x \to a^-} f(x) = f(a).$$

(Note that a function f is continuous at a if and only if it is both left-continuous and right continuous at a.)

f is continuous on the interval (a,b) if it is continuous at every point in the interval. That is, for every c with $a < c < b$

$$\lim_{x \to c} f(x) = f(c).$$

Some theorems:

Theorem 4 in §2.5:

If f and g are continoous at a, and c is some constant, then the following functions are also continuous at a:

- $f+g$
- $f-g$
- $c \cdot f$
- $f \cdot g$
- f/g, provided that $g(a) \neq 0$.

<u>Proof</u>: Use the limit laws we saw in Lecture 2. For example, to show that $h := f+g$ is continuous at a, we can use the Addition Law. Since $\lim_{x \to a} f(x) = f(a)$ and $\lim_{x \to a} g(x) = g(a)$ exist, we have that

$$\lim_{x \to a} h(x) = \lim_{x \to a} (f(x) + g(x))$$

$$= \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

$$= f(a) + g(a) = h(a).$$

Since $\lim_{x \to a} h(x) = h(a)$, we conclude that the function $h = f+g$ is continuous at a. $\square$

Theorem 5 in § 2.5:

(a) Polynomial functions are continuous at every point in $(-\infty, \infty) = \mathbb{R}$.

(b) Any rational function is continuous on every point of its domain.

Theorem 7 in § 2.5:

The following types of functions are continuous at every point of their domain:

- Polynomials. e.g. $x^{100} + 2x^{50} + x - 5$
- Rational functions. e.g. $\frac{x^3 + x}{1 + 2x}$
- Root functions. e.g. $\sqrt[3]{x}$
- Trigonometric functions. e.g. $\sin x, \cos x, \tan x, \dots$
- Inverse trigonometric functions. $\sin^{-1} x, \cos^{-1} x, \tan^{-1} x, \dots$
- Exponential functions. e.g. $2^x, 3^x, e^x, \dots$
- Logarithmic functions. e.g. $\log_2 x, \log_3 x, \log x, \dots$

Theorem 9 in § 2.5:

The composition of continuous functions is continuous on their domain.

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right).$$

when f is continuous at $b = g(a)$.

Example: Where is the function

$$f(x) = \frac{\ln x + \tan^{-1} x}{x^2 - 1}$$

continuous?

<u>Solution:</u>

- We know that $\ln x$ is continuous everywhere on its domain, whic is

$$\text{Domain}(\ln x) = (0, \infty).$$

- Similarly for $\tan^{-1} x$, whose domain is

$$\text{Domain}(\tan^{-1} x) = \mathbb{R} = (-\infty, \infty).$$

- Thus, we have that

$$\text{Domain}(\ln x + \tan^{-1} x) = (0, \infty).$$

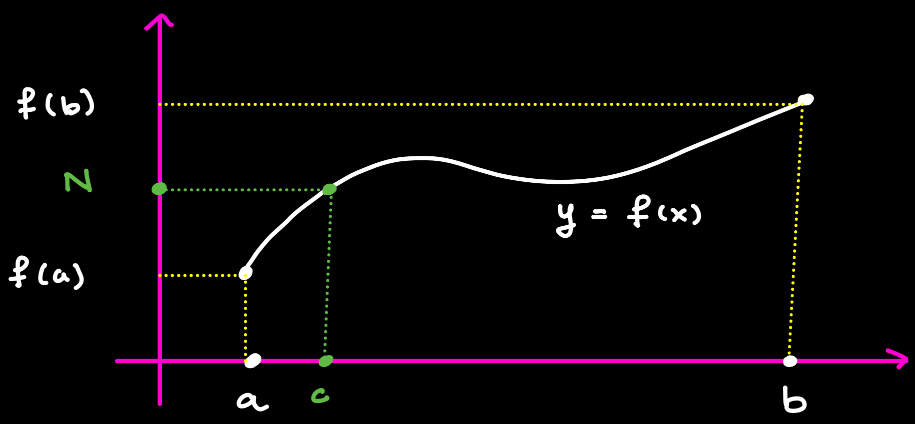
- The function $x^2 - 1$ is a polynomial, so it is continuous everywhere.

  Theorem 4 tells us that f is continuous at every point $a > 0$ for which $a^2 - 1 \neq 0$.

- We conclude that f is continuous on the intervals

$$(0, 1) \quad \text{and} \quad (1, \infty).$$

The Intermediate Value Theorem



The closed interval $[a,b]$ denotes the set of points r such that
$$a \le r \le b.$$
In particular, it includes a and b.

IVT: Supose that f is continuous at every point of the closed interval $[a,b]$, and let N be a point between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then, there exists a point $c \in [a,b]$ such that
$$f(c) = N.$$

Limits at ∞

Def: Let f be some function defined on some interval (a,∞). We write

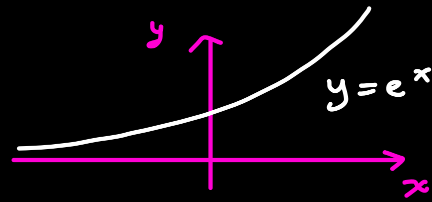
$$\lim_{x \to \infty} f(x) = L$$

to mean that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large.

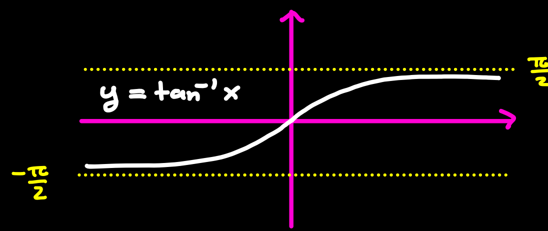
Similarly for limits to $-\infty$.

Examples:

- $\lim_{x \to -\infty} e^x = 0$.



- $\lim_{x \to \infty} \tan^{-1} x = \frac{\pi}{2}$.



Def: We say that f has a horizontal assymphote at the line $y = L$ if either

$$\lim_{x \to \infty} f(x) = L \quad \text{or} \quad \lim_{x \to -\infty} f(x) = L.$$