GALOIS GROUPS OF LOW DIMENSIONAL ABELIAN VARIETIES OVER FINITE FIELDS

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Abstract. We consider three isogeny invariants of abelian varieties over finite fields: the Galois group, Newton polygon, and the angle rank. Motivated by work of Dupuy, Kedlaya, and Zureick-Brown, we define a new invariant called the *weighted permutation representation* which encompasses all three of these invariants and use it to study the subtle relationships between them. We use this permutation representation to classify the triples of invariants that occur for abelian surfaces and simple abelian threefolds.

Contents

1. Introduction

The purpose of this article is to study the surprisingly subtle interactions between three isogeny invariants of abelian varieties over finite fields. We analyze which triples of invariants may occur and what restrictions they impose on the abelian variety in question. The interaction of these invariants is discussed in a letter from Serre to Ribet [Ser89, pp. 6] and has gained renewed interest following the publication of the database of abelian varieties over finite fields in the LMFDB [DKRV21b]. The availability of this data has demonstrated that the interaction between these invariants is more intricate than initially thought [DKRV21a], prompting the development of more refined invariants to better understand these relationships [DKZ24]. This article makes progress towards that goal. Even though this subject is interesting in its own right, it has applications to the Tate conjecture [Zar15, Zar22], monodromy groups of abelian varieties over number fields [Zyw22], Frobenius distributions [AS10, ABS24], and prime number races and Chebyshev biases in the context of function fields [BDKL24].

[0]November 7, 2024

By the Honda–Tate theorem [Tat66, Hon68, Tat71], the isogeny class of an abelian variety A is determined by its Frobenius polynomial, which is the characteristic polynomial of the Frobenius endomorphism acting on the ℓ-adic Tate module of A (where ℓ is a prime number which is not equal to the characteristic of the base field $\mathbf{F}_q$). All of our invariants are derived from the Frobenius polynomial; the first invariant is the Newton polygon of the Frobenius polynomial, which determines the p-adic valuations of the roots, the second invariant is the angle rank, which measures the nontrivial multiplicative relations between the roots of the Frobenius polynomial, and finally, we have the Galois group of the Frobenius polynomial as our third invariant. We classify triples of invariants that occur for abelian varieties of dimension ≤ 3; the dimension 3 case already exhibits some subtleties that should be expected in general.

In [DKZ24], the authors noticed that the Galois group, Newton polygon, and angle rank are not independent. The Galois group acts on the p-adic valuations of the roots (we visualize each root as a ball of radius proportional to its p-adic valuation) and this "weighted permutation representation" determines the Galois group, Newton polygon, and angle rank; see Definition 3.4. However, this does *not* imply that the Galois group and Newton polygon determine the angle rank. For example, the isogeny classes of abelian threefolds over $\mathbf{F}_2$ with LMFDB [LMF24] labels `3.2.ac_a_d` and `3.2.a_a_ad` have the same Newton polygon and Galois group, but different angle ranks. This example illustrates the necessity of considering more than just the isomorphism class of the Galois group.

The authors of [DKZ24] defined the *Newton hyperplane representation* of a geometrically simple abelian variety to encode this information. In this paper, we reinterpret the Newton hyperplane representation in terms of a *weighted permutation representation*. Moreover, we associate to *every* abelian variety $A/\mathbf{F}_q$ a weighted permutation representation. We then classify weighted permutation representations for abelian surfaces, and simple abelian threefolds.

1.1. **Statement of main results.** Let A be a g-dimensional abelian variety over a finite field $\mathbf{F}_q$ where q is a power of a prime number p. The *Frobenius polynomial* of A

$$P_A(T) := \det(T - \mathrm{Frob}_q \mid T_\ell A)$$

is the characteristic polynomial of the Frobenius endomorphism acting on the ℓ-adic Tate module $T_\ell A$. We define:

- the *Galois group of A*, which we denote G_A, to be the Galois group of the radical $h_A(T)$ of the Frobenius polynomial $P_A(T)$;
- the *q-Newton polygon of A* to be the Newton polygon of the Frobenius polynomial $P_A(T)$ with respect to the p-adic valuation ν on $\overline{\mathbf{Q}}_p$ normalised so that $\nu(q) = 1$; and
- the *angle rank of A*, which we denote δ_A, to be $\dim_{\mathbf{Q}} U_A \otimes \mathbf{Q}$ where $U_A \subset \overline{\mathbf{Q}}^\times$ is the subgroup generated by $\alpha/\sqrt{q}$ where α runs over all roots of $P_A(T)$.

Our main result is the classification of which triples of Galois group, Newton polygon, and angle rank occur for abelian surfaces and simple abelian threefolds. We do this by classifying weighted permutation representations (as defined in Section 3.1) of abelian surfaces and simple abelian

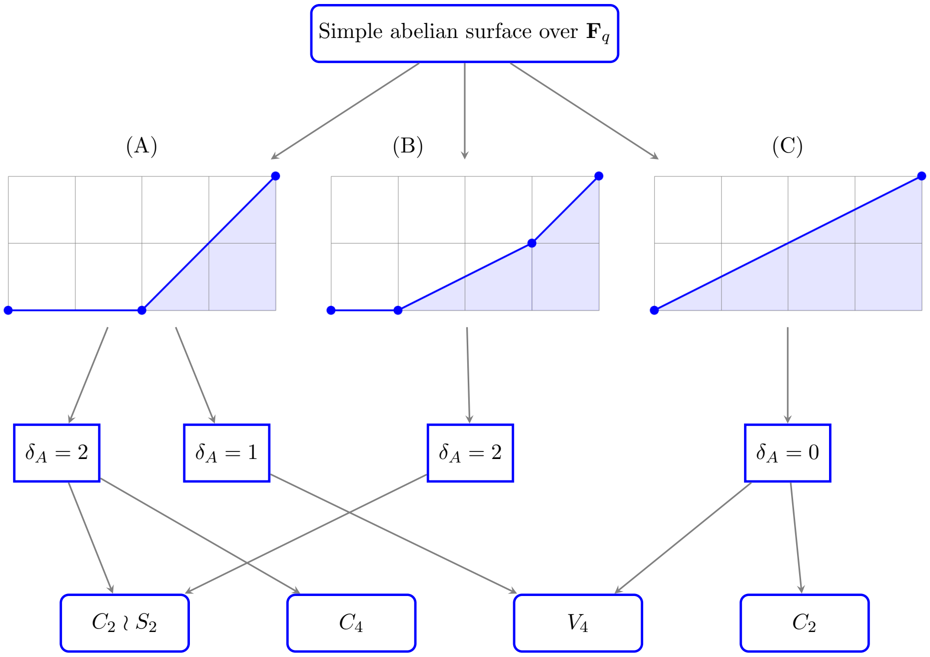

FIGURE 1.1. Possible isomorphism classes of Galois groups of simple abelian surfaces in terms of their Newton polygon, and angle rank δ_A.

threefolds. (The case $g = 1$ is easy and we discuss it in Section 4). The case of surfaces can be found in Theorem 5.1 and the case of threefolds can be found in Theorem 6.1.

The flowcharts in Figures 1.1–1.3 distill information from the tables in our main theorems. The purpose of these flowcharts is to serve as a "user's guide" to the tables in Theorem 5.1 and Theorem 6.1; in particular, if one has in hand an abelian surface or threefold, then the flowchart rules out certain Galois groups.

Corollary 1.1. *If A is an abelian surface, then the possible isomorphism classes of the Galois group G_A are determined by Figure 1.1 in the simple case, and by Figure 1.2 in the nonsimple case. Moreover, every possibility occurs.*

Corollary 1.2. *If A is a simple abelian threefold, then the possible isomorphism classes of the Galois group G_A are determined by Figure 1.3. Moreover, every possibility occurs.*

1.2. **Outline.** In Section 2 we introduce background and notation used throughout this paper. A reader familiar with abelian varieties may choose to skip directly to Section 3, where we introduce a key tool in our paper, the weighted permutation representation. In Section 4 we warm up by classifying permutation representations of elliptic curves. In Section 5 and Section 6 we classify weighted permutation representations of abelian surfaces and simple abelian threefolds respectively. We finish by listing further inverse Galois questions in Section 7.

The code associated to this article is written in `Magma` [BCP97] and is publically available from the GITHUB repository [AFV].

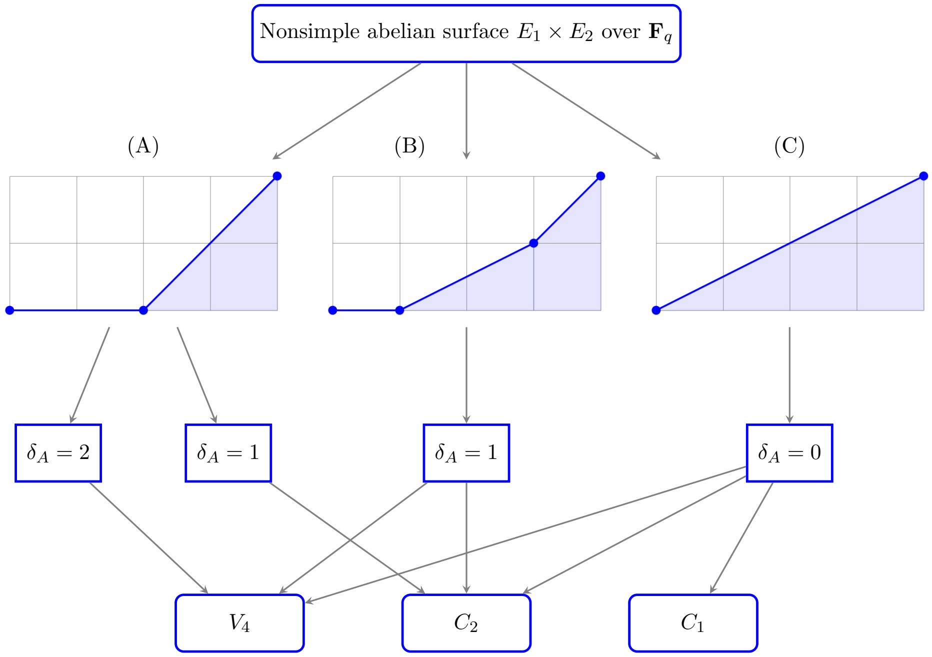

FIGURE 1.2. Possible isomorphism classes of Galois groups of simple abelian surfaces in terms of their Newton polygon, and angle rank δ_A.

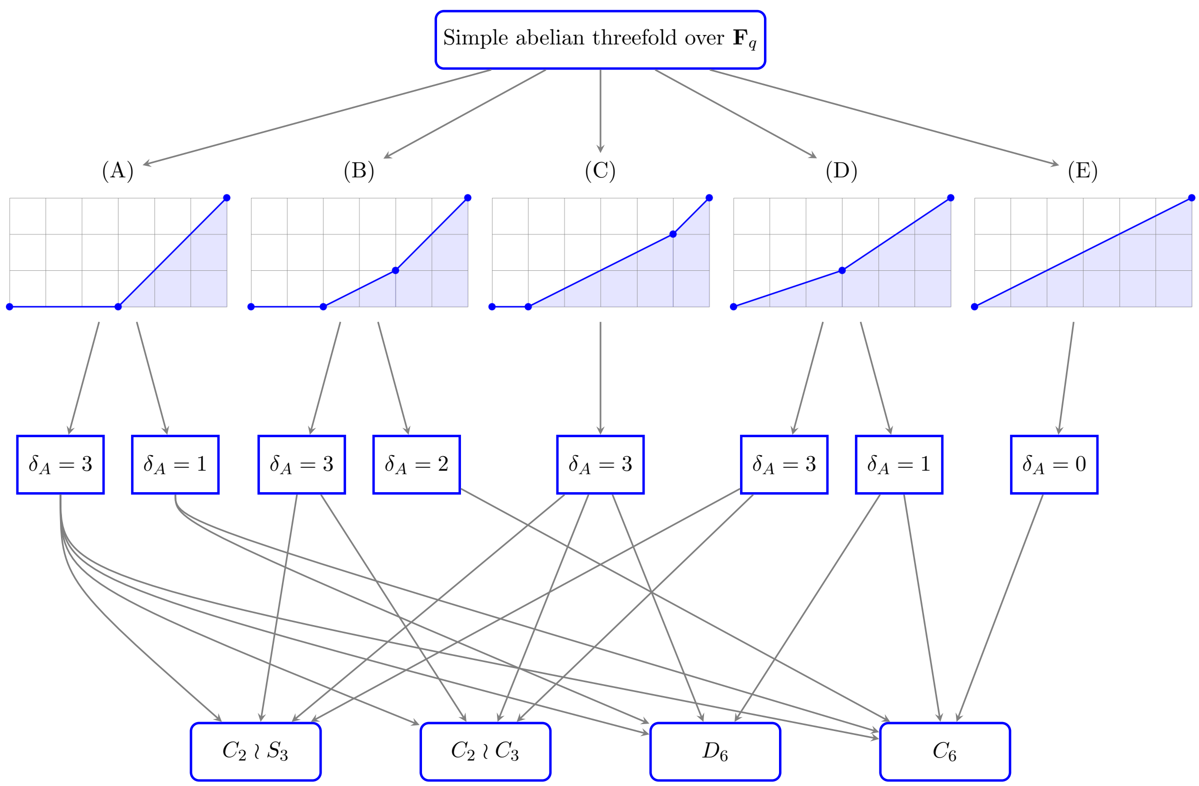

FIGURE 1.3. Possible isomorphism classes of Galois groups of simple abelian threefolds, in terms of their Newton polygon and angle rank δ_A.

1.3. **Acknowledgements.** We thank Deewang Bhamidipati, Soumya Sankar, John Voight, and David Zureick-Brown for helpful conversations about this project. We also thank Everett Howe for several useful correspondences. SF was supported by the Woolf Fisher and Cambridge Trusts through a Woolf Fisher scholarship and by Céline Maistret's Royal Society Dorothy Hodgkin Fellowship. SV was supported by the National Science Foundation under grant number DMS2303211.

2. Background and notation

2.1. **Honda–Tate theory.** Let $A/\mathbf{F}_q$ be an abelian variety. A celebrated theorem of Honda and Tate classifies isogeny classes of abelian varieties over finite fields. Let $g := \dim(A) > 0$ and recall that $P_A(T) \in \mathbf{Z}[T]$ is the characteristic polynomial of Frobenius.

The roots of $P_A(T)$ have absolute value $\sqrt{q}$ in all their complex embeddings; an algebraic integer with this property is called *q-Weil number*. A *q-Weil polynomial* is a monic integral polynomial whose roots are all q-Weil numbers. Fix an algebraic closure $\overline{\mathbf{Q}}$ of $\mathbf{Q}$ inside of $\mathbf{C}$, and an embedding $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$. We write ν for the p-adic valuation on $\overline{\mathbf{Q}}_p$, normalized so that $\nu(q) = 1$.

The statement below is the one presented in [Poo06, Theorem 4.2.12].

Theorem 2.1 (Honda–Tate Theorem)**.**

(1) If A is a simple abelian variety, then $P_A(T) = h_A(T)^e$ for some irreducible polynomial $h_A(T) \in \mathbb{Z}[T]$ and some $e \geq 1$.

(2) There is a bijection between isogeny classes of simple abelian varieties over $\mathbb{F}_q$ and conjugacy classes of q-Weil numbers.

(3) Given $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$-conjugacy class of q-Weil numbers, let $h_A(T)$ be the minimal polynomial of any element of this conjugacy class. Then there exists a unique integer $e_A := e \geq 1$ such that $h_A(T)^e = P_A(T)$ for some simple abelian variety A over $\mathbf{F}_q$. Moreover, e is the smallest positive integer such that:

   (a) $h_A(0)^e > 0$, and

   (b) For each monic $\mathbf{Q}_p$-irreducible factor $g(T) \in \mathbf{Q}_p[T]$ of $h_A(T)$, the valuation $\nu(g(0)^e)$ is in $\mathbf{Z}$.

The polynomial $h_A(T)$ is the *minimal polynomial of the q-Frobenius endomorphism of A*. When $P_A(T)$ is totally complex (i.e., has no real roots), the degree of $h_A(T)$ is equal to $2d$ for some positive integer d.

In general, the isogeny factorization of A yields a factorization of the Frobenius polynomial. In particular, by the Honda–Tate theorem, two abelian varieties A and B are isogenous if and only if $P_A(T)$ is equal to $P_B(T)$.

This observation is sufficient to understand the classification of our three isogeny invariants in the case of elliptic curves (see Section 4). We now proceed to give more background that will be useful for higher dimensional abelian varieties.

We will use the following as a running example throughout the article to consolidate our definitions and notations. This example appears in [Shi82, Example 6.1] and [Zyw22, Example 1.7] (see also [GGL24, Example 4.2.9]).

Example 2.2 (Shioda's example)**.** Let $q = p = 19$, and let A be the Jacobian of the hyperelliptic curve C with affine equation $y^2 = x^9 - 1$, defined over the field $\mathbf{F}_{19}$. The curve C has genus $g = 4$ and therefore A is an abelian fourfold. By calculating $\#C(\mathbf{F}_{19^r})$ for $r = 1, 2, 3, 4$, we are able to estimate the zeta function of C to enough precision to recover the Frobenius polynomial $P_A(T)$. It is given by:

$$P_A(T) = T^8 + 8T^7 + 28T^6 + 8T^5 - 170T^4 + 152T^3 + 10108T^2 + 54872T + 130321.$$

This polynomial factors as $P_A(T) = P_E(T)P_B(T)$, where E is the elliptic curve $y^2 = x^3 - 1$ in the isogeny class `1.19.i` with Frobenius polynomial $P_E(T) = T^2 + 8T + 19$, and B is an abelian threefold in the isogeny class `3.19.a_j_acm` with Frobenius polynomial $P_B(T) = T^6 + 9T^4 - 64T^3 + 171T^2 + 6859$. By the Honda–Tate theorem A is isogenous to the product $E \times B$.

2.2. **Newton polygons of Frobenius polynomials.** Let A be a g-dimensional abelian variety over $\mathbf{F}_q$. At this moment, we do not assume that A is simple.

Definition 2.3 (q-Newton polygon)**.** The *q-Newton polygon* of A is the ν-adic Newton polygon of $P_A(T)$. More precisely, if $P_A(T) = \sum_{j=0}^{2g} a_{2g-j}T^j$, then the q-Newton polygon of A is the lower convex hull of the finite set

$$\{(j, \nu(a_j)) : 0 \le j \le 2g, \text{ and } a_j \ne 0\} \subset \mathbf{R}^2.$$

Example 2.4. Continuing with Example 2.2, we note that both E and B are ordinary varieties. Since $A \sim E \times B$, the Newton polygon of A is also ordinary, and it is obtained by concatenating those of E and B. Alternatively, one can notice that -170 (the middle coefficient of $P_A(T)$) is not divisible by $p = 19$.

2.3. **Angle rank of Frobenius polynomials.** We recall the following definition from [DKRV21b, DKZ24, ABS24].

Definition 2.5. Consider the multiplicative subgroup $U_A \subset \overline{\mathbf{Q}}^\times$ generated by the normalized Frobenius eigenvalues $u \coloneqq \alpha/\sqrt{q}$ where α ranges over the roots of $h_A(T)$. The *angle rank of A* is denoted δ_A and is defined to be dimension of $U_A \otimes \mathbf{Q}$.

2.4. **Galois groups of Frobenius polynomials.** Given an abelian variety A, denote by R_A the *set of roots*, without multiplicity, of the Frobenius polynomial $P_A(T)$.

Definition 2.6 (Galois group)**.** The *Galois group* of A is the Galois group of the minimal polynomial of Frobenius $h_A(T)$. Equivalently, G_A is the largest quotient of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ over which the permutation action on R_A factors.

Definition 2.7. In the case that A is simple, we will denote by K_A the center of the endomorphism algebra $\mathrm{End}(A) \otimes \mathbf{Q}$. We have that $K_A \cong \mathbf{Q}[T]/(h_A(T))$ is a number field.

For "most" abelian varieties the polynomial $P_A(T)$ is totally complex (i.e., has no real roots), and K_A is a *complex multiplication* (CM) number field. See [Dod84] for some background on Galois groups of CM number fields, and [DKZ24, Section 2.2] for a discussion on Galois groups of q-Weil polynomials.

Example 2.8. Continuing with Example 2.2, we have that the splitting field of $P_A(T)$ is the degree 6 field $\mathbf{Q}(\zeta_9)$, where ζ_9 is a primitive 9^{th}-root of unity. This already implies that the 8 roots of $P_A(T)$ are algebraically dependent. Recalling that $A \sim E \times B$, observe that $K_E = \mathbf{Q}(\sqrt{-3})$, which is contained in $K_B = \mathbf{Q}(\zeta_9)$. Note that G_A is a permutation group acting on the 8 element set R_A. But as abstract groups, $G_A \cong \mathrm{Gal}(\mathbf{Q}(\zeta_9)/\mathbf{Q}) \cong C_6$.

Denote by $R_E = \{\eta, 19\eta^{-1}\}$ and $R_B = \{\alpha, \beta, \gamma, 19\alpha^{-1}, 19\beta^{-1}, 19\gamma^{-1}\}$ the sets of roots of $P_E(T)$ and $P_B(T)$ respectively. For an appropriate such labelling, we have

$$\eta = \frac{\alpha \cdot \beta \cdot \gamma}{19}. \tag{2.1}$$

In Example 3.9 that the $\mathbf{Q}$-vector space $U_A \otimes \mathbf{Q}$ has dimension 3, i.e., $\delta_A = 3$.

Instead of viewing G_A as a permutation subgroup of S_n for $n = \#R_A$, it will be convenient to use a "CM specific" permutation representation.

2.5. **The group of signed permutations.** We now describe the abstract group which Galois groups of totally complex q-Weil numbers are naturally contained in. First note that when $h_A(T)$ is totally complex of degree $2d$ and the roots of $h_A(T)$ come in complex conjugate pairs. Moreover, the action of the Galois group G_A respects this partition.

Let X_{2d} be the set consisting of the symbols $1, \bar{1}, \ldots, d, \bar{d}$. Define W_{2d} to be the subgroup of $\mathrm{Sym}(X_{2d})$ which preserves the partition

$$X_{2d} = \{1, \bar{1}\} \sqcup \cdots \sqcup \{d, \bar{d}\}.$$

Upon a choice of labelling of roots, the Galois group of $h_A(T)$ may naturally be embedded in W_{2d}. We refer to the element $\iota := (1\bar{1}) \ldots (d\bar{d})$ as *complex conjugation*.

2.5.1. *Subgroup labelling.* We now briefly describe our naming conventions for subgroups of W_{2d}. A group H is denoted `G.d.t.letter.k` if it is isomorphic to G, contained in W_{2d}, and acts transitively on X_{2d}. The W_{2d} conjugacy class is indexed by `letter`, and the groups in that conjugacy class are indexed by the tiebreaker `k` (a positive integer). The use of `nt` instead of `t` indicates that H acts intransitively on X_{2d}. For example, the group `C2.4.nt.c.1` refers to a group which is isomorphic to C_2, contained in W_4, and is intransitive. The label `c` refers to the conjugacy class in W_4 and the index `1` means that it is the first listed in its conjugacy class.

Remark 2.9. In the code associated to this article [AFV] we provide functions which compute and label the transitive subgroups of W_{2d} which contain complex conjugation in the file `src/W2d-subgroups.m`. Our labelling convention is well defined and essentially follows lexicographic ordering of the subgroups of the symmetric group $S_{2d} \supset W_{2d}$ (as described in [HL03]). See the file `src/subgroup-labelling.m`.

3. The weighted permutation representation

In this section we introduce a key tool in this paper – the notion of a weighted permutation representation. This construction is heavily inspired by the definition of *Newton hyperpane arrangement* of Dupuy, Kedlaya, and Zureick-Brown [DKZ24].

3.1. **The weighted permutation representation.** We now describe the weighted permutation representation associated to an abelian variety, which is the central isogeny invariant in this article. It determines the Galois group, angle rank, and Newton polygon. The main purpose of our paper is to determine which weighted permutation representations occur from low dimensional abelian varieties. The weighted permutation representation is a reinterpretation of the *Newton hyperplane representation* discussed in [DKZ24].

Definition 3.1 (Weighted permutation representations)**.** Given a finite group G, a *weighted permutation representation of G* is a pair (w, ρ), where $w\colon X_{2d} \to \mathbf{Q}_{\geq 0}$ is a map of sets and $\rho\colon G \hookrightarrow W_{2d}$ is an inclusion of groups.

We say that a pair of weighted permutation representations (w, ρ) and (w, ρ') of G are *w-conjugate* if they are conjugate by an element of $\mathrm{Stab}(w)$, i.e., if there exists an element $\sigma \in W_{2d}$ such that $w = w \circ \sigma$ and $\rho' = \sigma^{-1}\rho(g)\sigma$ for all $g \in G$.

Definition 3.2 (Roots)**.** Recall that for an abelian variety A we write R_A for the set of roots of $h_A(T)$ without multiplicity. We define $\mathcal{R}_A$ to be the *multiset* of roots of $h_A(T)$, that is:

1. When $h_A(T)$ is totally complex, $\mathcal{R}_A = R_A$.
2. When $h_A(T)$ has a real root α, it is counted with multiplicity two, and its duplicate is denoted by $\overline{\alpha}$.

We define $d_A = \#\mathcal{R}_A/2$.

Definition 3.3 (Indexing and weighting)**.** An *indexing of roots* of $h_A(T)$ is a bijection $\mathcal{I}\colon X_{2d} \to \mathcal{R}_A$ which satisfies the following conditions:

1. $\mathcal{I}$ respects complex conjugation, i.e., $\mathcal{I}(\overline{k}) = \overline{\mathcal{I}(k)}$ for each $1 \leq k \leq d$, and
2. the indices climb the Newton polygon, i.e.,

$$\nu(\alpha_i) \leq \nu(\alpha_j) \leq \nu(\overline{\alpha}_j) \leq \nu(\overline{\alpha}_i)$$

   for each pair of indices $1 \leq i \leq j \leq d$.

Any indexing of the roots naturally gives a *weighting* $w_A\colon X_{2d} \to \mathbf{Q}_{\geq 0}$ given by $k \mapsto \nu(\alpha_k)$ and $\bar{k} \mapsto \nu(\overline{\alpha}_k)$ for $k \in \{1, \ldots, d\}$. We omit the choice of indexing from the notation, since every choice of indexing yields the same weighting w_A. Note that the weighting w_A associated to an abelian variety A is uniquely determined by the q-Newton polygon of A.

Definition 3.4 (Weighted permutation representation, totally complex case)**.** Suppose $h_A(T)$ is totally complex. Given an indexing $\mathcal{I}$, we obtain representation

$$\rho_{\mathcal{I}}\colon G_A \hookrightarrow \mathrm{Sym}(X_{2d}) \cong S_{2d}$$

whose image lies in W_{2d}. The pair $(w_A, \rho_{\mathcal{I}})$ is the *weighted permutation representation* associated to A with respect to the indexing $\mathcal{I}$.

The definition above *canonically* defines the weighted permutation representation associated to A up to w_A-conjugacy when $P_A(T)$ is totally complex. The following definition gives a notion of weighted permutation representation when $P_A(T)$ is totally real; although the definition does not appear canonical, we will show later that it is in fact unique up to isomorphism.

The simple isogeny classes of abelian varieties with real eigenvalues are highly constrained.

Lemma 3.5 ([Wat69, pp. 528])**.** *Let A be a simple abelian variety whose Frobenius eigenvalues are real, then:*

(1) If q is a square, A is a supersingular elliptic curve, and either $h_A(T) = (T \pm \sqrt{q})$ for some choice of sign, or

(2) If q is not a square, A is a supersingular abelian surface, and $h_A(T) = (T^2 - q)$.

Moreover, in case (1) we have $\mathcal{R}_A = \{\sqrt{q}, \overline{\sqrt{q}}\}$ or $\{-\sqrt{q}, -\overline{\sqrt{q}}\}$, and in case (2) we have $\mathcal{R}_A = \{\sqrt{q}, \overline{\sqrt{q}}, -\sqrt{q}, -\overline{\sqrt{q}}\}$.

Definition 3.6 (Weighted permutation representation, totally real case)**.** Suppose that A has only real Frobenius eigenvalues.

(1a) If q is a square and A has one eigenvalue, then G_A is trivial. Define $\rho_{\mathcal{I}} \colon G_A \to W_2$ to be the trivial map for all indexings.

(1b) If q is a square and A has two eigenvalues, then G_A is trivial. Define $\rho_{\mathcal{I}} \colon G_A \to W_4$ to be the trivial map for all indexings.

(2) If q is not a square, then $G_A \cong C_2$. Define $\rho_{\mathcal{I}} \colon G_A \to W_4$ to be the homomorphism sending the nontrivial element on G_A to $(12)(\bar{1}\bar{2})$ for all indexings.

The pair $(w_A, \rho_{\mathcal{I}})$ is the *weighted permutation representation* associated to A with respect to the indexing $\mathcal{I}$.

We now define the weighted permutation representation associated to an abelian variety A for which $h_A(T)$ need not be totally real or totally complex; it is essentially the direct sum of its totally real and totally complex parts. In this case A is isogenous to a product $B \times C$ where $h_B(T)$ is totally real and $h_C(T)$ is totally complex. Let (w_B, ρ_B) and (w_C, ρ_C) be the weighted permutation representations corresponding to the factors B and C respectively. Let $(w_B \oplus w_C, \rho_B \oplus \rho_C)$ be the direct sum of these weighted permutation representations; conjugate by W_{2d} so that the weight function is nondecreasing, i.e., so that

$$w(i) \leq w(j) \leq w(\bar{j}) \leq w(\bar{i})$$

for $i \leq j$. We define the weighted permutation representation associated to A to be the resulting weighted permutation representation. Lemma 3.7 follows by construction.

Lemma 3.7. *The weighted permutation representation associated to an abelian variety A is well defined up to w_A-conjugacy (irrespective of indexing). In particular, its image in W_{2d} is a subgroup $\mathcal{G}_A$ which is well-defined up to w_A-conjugacy.*

3.2. **The angle rank.** It turns out that the angle rank can be computed from the weighted permutation representation of an abelian variety in the following way.

Definition 3.8. Given a weighted permutation representation $(w, \rho\colon G \hookrightarrow W_{2d})$, define the *angle rank of (w, ρ)* to be $\mathrm{rk}(M) - 1$ where M is the $(d \times |G|)$-matrix whose i^{th}-column has entries $w(\sigma(i))$ where σ ranges over elements of G.

The angle rank of (w, ρ) is equal to the rank of the $(d \times |G|)$-matrix whose i^{th}-column has entries $w(\sigma(i)) - w(\sigma(\bar{i}))$, which is referred to as the Newton hyperplane matrix [DKZ24, Remark 3.4]. We show in Lemma 3.11 that the angle rank of an abelian variety is equal to the angle rank of its weighted permutation.

Example 3.9. Continuing with Example 2.8, fix a prime $\mathfrak{p}$ of the splitting field $K = \mathbf{Q}(\zeta_9)$ above $p = 19$, and let ν be the extension of the 19-adic valuation of $\mathbf{Q}$ extended to K. An indexing of the roots R_A according to Definition 3.3 is for example

$$\alpha_1 = \alpha, \quad \alpha_2 = \beta, \quad \alpha_3 = \eta, \quad \alpha_4 = \gamma, \tag{3.1}$$

if we ensure that $\nu(\alpha) = \nu(\beta) = \nu(\gamma) = \nu(\eta) = 0$. With this indexing, the weighted permutation representation $\rho\colon \operatorname{Gal}(\mathbf{Q}(\zeta_9)/\mathbf{Q}) \to W_8$ has image $H = \langle h \rangle$, where $h = (1\bar{2}\bar{4}\bar{1}24)(3\bar{3})$. With respect to this indexing, the multiplicative relation in Equation (2.1) becomes

$$\alpha_3 = \frac{\alpha_1\alpha_2\overline{\alpha}_4}{19} = \frac{\alpha_1\alpha_2}{\alpha_4}. \tag{3.2}$$

One can find this multiplicative relation by considering the Newton hyperplane matrix of this weighted permutation representation, and computing its kernel. By direct calculation we see that the rank of the Newton hyperplane matrix (which is equal to δ_A) is three.

3.3. **The divisor map.** Many of the proofs in this paper make use of the same key idea, which we elucidate here. Given a simple totally complex g-dimensional abelian variety A with Galois group G_A, we may construct the following. Let L be the Galois closure of the field K, i.e., the field generated by the roots of $P_A(T)$. Let $\mathcal{P}_L$ be the set of primes in L above p. Let $\mathbf{Q}\langle R_A\rangle$ denote the $\mathbf{Q}$-vector space of dimension $2d$ whose basis consists of the formal symbols $[\alpha]$ where α ranges over the set of Frobenius eigenvalues.

Let $\mathrm{Div}_{\mathbf{Q}}(\mathcal{O}_L)$ be the free $\mathbf{Q}$-module supported on the prime ideals of $\mathcal{O}_L$ and let $\mathrm{div}_A\colon \mathcal{O}_L \to \mathrm{Div}_{\mathbf{Q}}(\mathcal{O}_L)$ be the map sending each element $x \in \mathcal{O}_L$ to $\sum_{\mathfrak{p}} a_{\mathfrak{p}}\mathfrak{p}$ where $(x) = \prod_{\mathfrak{p}} \mathfrak{p}^{a_{\mathfrak{p}}}$ is the prime factorization of the ideal generated by x. By abuse of notation we write

$$\mathrm{div}_A\colon \mathbf{Q}\langle R_A\rangle \to \mathbf{Q}\langle \mathcal{P}_L\rangle$$

for the $\mathbf{Q}$-linear map given by linearly extending div_A on the roots $\alpha \in R_A$.

Note that $\mathbf{Q}\langle R_A\rangle$ naturally inherits the structure of a G_A-module from the action of G_A on R_A, and similarly $\mathbf{Q}\langle \mathcal{P}_L\rangle$ has an action of G_A inherited from the action of G_A on $\mathcal{P}_L$. The map div_A is G_A-equivariant. Note that div_A determines the permutation representation, and thus also determines the angle rank, Newton polygon, and Galois group.

It is natural to ask which G_A-module homomorphisms are permissible as the divisor map of a simple abelian variety. We list some necessary conditions which follow immediately from the construction.

Proposition 3.10. *Let A be a simple abelian variety with no real Frobenius eigenvalues. Then:*

(1) $\mathrm{div}_A([\alpha]+[\overline{\alpha}]) = \mathrm{div}_A(q)$ *for all* $\alpha \in R_A$*;*

(2) the G_A*-action on* R_A *is transitive;*

(3) the G_A*-action on* $\mathcal{P}_L$ *is transitive; and*

(4) if the Newton polygon has a segment of length m *which contains exactly two lattice points, then* $\#\mathcal{P}_L \mid \frac{1}{m}\#G_A$*.*

We now quickly use the divisor map to show the following lemma.

Lemma 3.11. *The angle rank* δ_A *of an abelian variety is equal to the angle rank of its weighted permutation representation. Moreover,* $\delta_A = \mathrm{rk}(\mathrm{div}_A) - 1$*.*

Proof. Consider the subspace $V_p = \mathrm{span}(\mathrm{div}_A(p)) \subset \mathbf{Q}\langle \mathcal{P}_L\rangle$. We first claim that the angle rank of an abelian variety is precisely the rank of the linear map $\mathbf{Q}\langle R_A\rangle \to \mathbf{Q}\langle \mathcal{P}_L\rangle / V_p$ induced by div_A. Clearly a multiplicative relation between the Frobenius eigenvalues gives rise to an element in the kernel of this map. Conversely, given an element $\sum_{i=1}^{2g} k_i\alpha_i$ in the kernel, we have

$$\left(\prod_i \alpha_i^{k_i}\right) = (q)^{\sum k_i/2},$$

so there exists $u \in \mathcal{O}_L^\times$ such that

$$\prod_i q^{-k_i/2}\alpha_i^{k_i} = u.$$

Because u has length 1 in all complex embeddings, u is a root of unity. Now observe that the rank of $\mathbf{Q}\langle R_A\rangle \to \mathbf{Q}\langle \mathcal{P}_L\rangle / V_p$ is precisely one less than the rank of $\mathrm{div}_A \colon \mathbf{Q}\langle R_A\rangle \to \mathbf{Q}\langle \mathcal{P}_L\rangle$. After removing duplicate rows and columns, the matrix representing the latter is precisely the matrix given in the definition of the angle rank of a weighted permutation representation. □

Observe that every such map $\mathbf{Q}\langle X_{2d}\rangle \to \mathbf{Q}^\ell$ satisfying the properties above gives rise to a weighted permutation representation. To prove that a certain weighted representation cannot occur, it suffices to show that there does not exist a lift $\mathbf{Q}^{2d} \to \mathbf{Q}^\ell$ satisfying the conditions above.

4. Warm-up: elliptic curves

We begin with the case when A is an elliptic curve. In accordance with the labelling convention described in Section 2.5 we write `W2.2.t.a.1` and `C1.2.t.a.1` for the subgroups of order 2 and 1 of W_2. The following proposition is immediate.

Proposition 4.1. *Let* A *be an elliptic curve. Then the image,* $\mathcal{G}_A$*, of the weighted permutation representation associated to* A *is equal to* W_2 *except when* q *is a square and* $P_A(T) = (T \pm \sqrt{q})^2$*, in which case* A *is supersingular and* $\mathcal{G}_A = \{\mathrm{id}\}$*. More precisely, the possible combinations of Galois group, Newton polygon, and angle rank that occur are displayed in Tables 4.1 and 4.2.*

w_A-conjugacy class	Angle rank	Occurs	Example(s)
W2.2.t.a.1	1	Yes	1.2.ab
C1.2.nt.a.1	0	No	

TABLE 4.1. The w_A-conjugacy classes of subgroups $G \subset W_2$ which occur as the image of the weighted permutation representation associated to an ordinary elliptic curve.

w_A-conjugacy class	Angle rank	Occurs	Example(s)
W2.2.t.a.1	0	Yes	1.2.ac
C1.2.nt.a.1	0	Yes	1.4.ae

TABLE 4.2. The w_A-conjugacy classes of subgroups $G \subset W_2$ which occur as the image of the weighted permutation representation associated to a supersingular elliptic curve.

5. Abelian surfaces

In this section we prove the following theorem.

Theorem 5.1. *Let A be an abelian surface.*

- ***Permutation representations in W_4:*** *If A has permutation representation contained in W_4, then the possible images $\mathcal{G}_A$ of the weighted permutation representation associated to A are given in Tables A.2 and A.5 when A is ordinary, in Tables A.3 and A.6 when A is almost ordinary, and in Tables A.4 and A.7 when A is supersingular. The permutation representation determines whether A is geometrically simple; this information can be seen in the tables as well.*
- ***Permutation representations in W_2 when A is simple:*** *If A has a permutation representation contained in W_2 and is simple, then A is supersingular and hence the angle rank is 0. Both trivial Galois group and Galois group C_2 occur. A is not geometrically simple.*
- ***Permutation representations in W_2 when A is not simple:*** *Now suppose A has a permutation representation contained in W_2 and is not simple. Then A is isogenous (over $\mathbf{F}_q$) to E^2 for some elliptic curve E then the weighted permutation representation associated to A takes values in W_2, and its image is equal to that associated to E (and classified in Proposition 4.1).*

The rest of the section is dedicated to proving Theorem 5.1. The third point follows from the definition and the section on elliptic curves. The second point follows from the classification given in [Xin94]; see also [MN02, Theorem 2.9 (SS2)]. In the remainder of this section we assume A has permutation representation contained in W_4.

5.1. Permutation representations of subgroups of W_4.

The group W_4 is isomorphic to D_4, the dihedral group of order 8. The following lemma classifies the w-conjugacy classes of subgroups of W_4.

Lemma 5.2. *There are exactly 3 transitive and 7 intransitive subgroups of W_4 and each is recorded in Table A.1. Moreover, the only distinct subgroups which are w_A-conjugate are:*

(1) `C2.4.nt.b.1` *and* `C2.4.nt.b.2` *when A is either ordinary or supersingular, and*

(2) `C2.4.nt.c.1` *and* `C2.4.nt.c.2` *when A is supersingular.*

Each w_A-conjugacy class gives rise to an isomorphism class of permutation representation, and the angle ranks of these w_A-conjugacy classes $\mathcal{G} \subset W_4$ are recorded in Tables A.2–A.7.

Proof. The first claim then follows by a direct calculation. The angle ranks are computed using our implementation of Lemma 3.11 in the file `src/weighted-perm-rep.m` of our GitHub repository [AFV]. □

To prove Theorem 5.1 it suffices to show that:

(1) the cases we claim do not occur, actually do not occur; and

(2) in the cases that do occur, the permutation representation determines whether the abelian surface is geometrically simple.

In the remaining cases, we provide an example in Tables A.2–A.7, which realizes the given permutation representation. We remark that the angle ranks displayed in the LMFDB are numerical approximations, but we verify these examples explicitly via a slower (but deterministic) algorithm; see the file `tables/verify-angle-rank.m` in our GitHub repository [AFV].

5.2. **Proof of Theorem 5.1 in the simple case.** For a simple abelian surface, the permutation representation acts transitively except when A is supersingular with real Frobenius eigenvalues.

5.2.1. *The ordinary case.* In this case, every possible transitive permutation representation occurs. Therefore, to prove the theorem in this case, it suffices to show that a simple ordinary abelian surface is not geometrically simple if and only if its permutation representation is `V4.4.t.a.1`.

Proposition 5.3. *A simple abelian variety has a unique simple factor over every finite extension of the base field. In particular, if A is simple of prime dimension and it splits over a finite extension of $\mathbf{F}_q$, then it does so as the power of an elliptic curve.*

Proof. This follows immediately from [CCO14, Proposition 1.2.6.1]. □

Corollary 5.4. *If A is a simple abelian variety of prime dimension which is not geometrically simple and not supersingular, then A is ordinary and has angle rank 1.*

Proof. Since A is simple but not geometrically simple by Proposition 5.3 there exists an extension $\mathbf{F}_{q^k}$ of $\mathbf{F}_q$ over which A becomes isogenous to E^g for some ordinary elliptic curve $E/\mathbf{F}_{q^k}$ (in particular A is ordinary). Angle rank is invariant under base change, so it follows that $\delta_A = 1$. □

Lemma 5.5. *An ordinary geometrically simple abelian surface A has angle rank 2.*

Proof. We prove the contrapositive. Assume that $\delta_A < 2$. This implies that there exists a multiplicative relation among the normalized Frobenius eigenvalues $u_1^{r_1} u_2^{r_2} = 1$. Note that $r_1 r_2 = 0$ implies that some u_i is a root of unity, which would imply that A is not ordinary. Thus, we have that both r_1 and r_2 are nonzero integers. Let α_1 and α_2 be the two Frobenius eigenvalues with

$\nu(\alpha_1) = \nu(\alpha_2) = 0$. From the multiplicative relation we deduce that $r_1 = -r_2$, so that $(u_1/u_2)^{r_1} = 1$ and $\alpha_2 = \zeta_k \alpha_1$ where ζ_k is a k^{th} root of unity. The Frobenius eigenvalues of the base change of A to $\mathbf{F}_{q^k}$ are precisely the k-th powers of the Frobenius eigenvalues of A. Because $\alpha_1^k = \alpha_2^k$, the Honda–Tate theorem implies that the base change of A to $\mathbf{F}_{q^k}$ is isogenous to the square of an elliptic curve, so A is not geometrically simple. □

Lemma 5.6. *Let A be a simple ordinary abelian surface. Then, exactly one of the following conditions holds.*

(1) A is geometrically simple and $G_A \cong C_4$ or W_4.

(2) A is not geometrically simple and $G_A \cong V_4$.

Proof. By Lemma 5.5 if A is geometrically simple then A has angle rank 2, and by Corollary 5.4 if A is not geometrically simple the it has angle rank 1. The claim follows from the angle ranks computed in Lemma 5.2 (and recorded in Table A.2). □

5.2.2. *The almost ordinary case.* Every simple almost ordinary abelian surface is geometrically simple by Corollary 5.4, so the following lemma completes the proof.

Lemma 5.7. *A simple almost ordinary abelian surface has Galois group W_4.*

Proof. Suppose for the sake of contradiction that A is an almost ordinary abelian variety with Galois group C_4 or V_4. Let div_A be the corresponding divisor map as discussed in Proposition 3.10. By point (4), we have $\#\mathcal{P}_L \mid 2$, where $\mathcal{P}_L$ is the set of primes above p in L. From Lemma 3.11, we have $\delta_A = \text{rk}(\text{div}_A) - 1 \leq \#\mathcal{P}_L - 1$, which contradicts Table A.3. □

Remark 5.8. One can also see Lemma 5.7 as follows: by the theory of Newton polygons and the Honda–Tate theorem, $P_A(T)$ has an irreducible linear and quadratic factor over $\mathbf{Q}_p$, so the quartic extension $K/\mathbf{Q}$ is not Galois.

5.2.3. *The supersingular case.* It follows from the Honda–Tate theorem that every supersingular abelian variety is geometrically isogenous to the power of an elliptic curve. The proof is concluded with the following well known result which shows that no supersingular abelian variety of dimension $g > 1$ can have Galois group W_{2g}.

Lemma 5.9. *Suppose A has Galois group $G_A \cong W_{2d}$. Then either:*

(1) A is a power of a supersingular elliptic curve E with $P_E(T) \neq (T \pm \sqrt{q})^2$, or

(2) A has angle rank $\delta_A = d$.

Proof. Suppose that A is not supersingular. Then, every normalized Frobenius eigenvalue $u = \alpha/\sqrt{q}$ satisfies $\nu(u) \neq 0$. Fix an indexing $\mathcal{I}\colon X_{2d} \to R_A$ and consider the vector $\mathbf{v} := (v_1, \dots, v_d)$ with nonzero entries $v_j := \nu(u_j)$. Acting on $\mathbf{v}$ by the transpositions $(k\bar{k}) \in W_{2d}$, we see that the Newton hyperplane matrix contains the $(d \times d)$-minor

$$\begin{bmatrix} v_1 & v_1 & \cdots & v_1 \\ v_2 & -v_2 & \cdots & v_2 \\ \vdots & \vdots & \ddots & \vdots \\ v_d & v_d & \cdots & -v_d \end{bmatrix},$$

which is visibly similar to the diagonal matrix $\mathrm{diag}(v_1, \ldots, v_d)$. □

5.3. **Proof of Theorem 5.1 in the nonsimple case.** If A is isogenous to the square of an elliptic curve, then the claim follows immediately from Proposition 4.1. Therefore suppose that A is isogenous over $\mathbf{F}_q$ to a product $E_1 \times E_2$ of non-isogenous elliptic curves. Note that when A is supersingular, there is nothing to show so it suffices to consider the ordinary and almost ordinary cases. Let α_i and $\overline{\alpha}_i$ be the Frobenius eigenvalues of E_i for each $i = 1, 2$.

5.3.1. *Nonsimple ordinary abelian surfaces.* In this case, both E_1 and E_2 are ordinary elliptic curves and the Frobenius eigenvalues $\alpha_i, \overline{\alpha}_i$ are not real, and therefore $\mathcal{G}_A$ contains complex conjugation. It is now easy to see that the Galois group is `V4.4.nt.a.1` if $\mathbf{Q}(\alpha_1) \neq \mathbf{Q}(\alpha_2)$ and `C2.4.nt.a.1` otherwise.

Remark 5.10. In fact, using Lemma 5.3 of [KS00], it is possible to show that in this case, the two elliptic curves are geometrically isogeneous if and only if the Galois group is `C2.4.nt.a.1`, but we will not need this.

5.3.2. *Nonsimple almost ordinary abelian surfaces.* In this case we may assume without loss of generality that E_1 is ordinary and E_2 is supersingular. First note that $\mathcal{G}_A$ cannot be the w_A-conjugate to `C2.4.nt.b.1` or `C1.4.nt.a.1` since in this case the Frobenius eigenvalues of E_1 are fixed by G_A, which cannot occur.

Lemma 5.11. *Let E_1 and E_2 be elliptic curves over $\mathbf{F}_q$. Suppose that E_1 is ordinary and E_2 is supersingular. Then, the corresponding number fields satisfy $K_1 \cap K_2 = \mathbf{Q}$.*

Proof. If $K_2 = \mathbf{Q}$ there is nothing to show. Suppose that this is not the case, and assume in search of a contradiction that $K_1 = K_2 = K$. Then $\alpha_1 = u\alpha_2$ for $u \in K$ of absolute value 1. Note that u can't be a root of unity, since that would imply that α_1 is a supersingular q-Weil number. Since α_2 is a supersingular q-Weil number, some power of u is an *algebraic integer*. This implies that u is also an algebraic integer which has length 1 in all complex embeddings, and thus $u \in \mathcal{O}_K^\times$. But K is quadratic imaginary, implying that u is a root of unity, a contradiction. □

6. Abelian threefolds

Theorem 6.1. *Let A be a simple abelian threefold.*

- ***Permutation representations in*** W_6. *If A has permutation representation contained in W_6, then the possible images of the weighted permutation representation associated to a simple abelian threefold is given in Tables A.8–A.12. Each table corresponds to one Newton polygon. The permutation representation determines whether A is geometrically simple or not, and this information can be found in the tables.*
- ***Permutation representations in*** W_2. *If the permutation representation of A is not contained in W_6, then it is contained in W_2. In this case $e_A = 1$, where e_A is as in the statement of the Honda Tate–theorem. Such abelian varieties have Galois group C_2, angle rank 1, and Newton polygon type (D). They are geometrically simple.*

The second point follows from [Xin94]; see [ABS24, Theorem 6.1.1, Lemma 6.1.2] for more detail. In the remainder of this section we assume all abelian threefolds in question are simple and have permutation representation contained in W_6. We first classify the permutation representations that occur, and then in Section 6.7 classify whether permutation representations that occur are geometrically simple or not. The rest of this section is dedicated to proving Theorem 6.1.

6.1. **Signed permutations on three elements.** The following lemma follows by a direct calculation in `Magma`. See the file `src/W2d-subgroups.m` in our GitHub repository [AFV].

Lemma 6.2. *There are exactly* 10 *transitive subgroups in* W_6 *which contain the complex conjugation element* $\iota \in W_6$*, and each is listed in Table A.1. These* 10 *subgroups are contained in exactly* 4 W_6*-conjugacy classes, namely those of:*

(1) W_6*,*
(2) $C_2 \wr C_3$ *(transitive label* `6T6`*) generated by* $(123)(\bar{1}\bar{2}\bar{3})$*,* $(1\bar{1})$*,* $(2\bar{2})$*, and* $(3\bar{3})$*,*
(3) D_6 *generated by* $(123)(\bar{1}\bar{2}\bar{3})$ *and* $(12)(\bar{1}\bar{2})$*,*
(4) C_6 *generated by* $(123\bar{1}\bar{2}\bar{3})$*.*

Moreover, for each possible Newton polygon of an abelian threefold A*, the* w_A*-conjugacy classes of transitive subgroups of* W_6 *are recorded in Tables A.8–A.12.*

Similarly to the case of surfaces, to prove Theorem 6.1 it suffices to show that:

(1) the cases we claim do not occur, actually do not occur; and
(2) in the cases that do occur, the permutation representation determines whether the abelian surface is geometrically simple (this is in Section 6.7).

In the remaining cases, we provide an example in Tables A.8–A.12 which realizes the given permutation representation. We now prove (1).

6.2. **Proof of Theorem 6.1 in the ordinary case.** In this case, the table shows that every possible permutation representation occurs, so there is nothing to prove.

6.3. **Proof of Theorem 6.1 in the almost ordinary case.** We first classify permutation representations that occur, and then give proofs of geometric simplicity. The following two lemmas complete the classification of permutation representations in the case of simple almost ordinary abelian threefolds.

Lemma 6.3. *An almost ordinary abelian threefold cannot have Galois group* C_6*.*

Proof. Suppose for the sake of contradiction that A is an almost ordinary abelian threefold with Galois group $G_A \cong C_6$. Note that all permutation representations of C_6 are conjugate in the almost ordinary case, so it suffices to show that the image of the permutation representation is not conjugate to `C6.6.t.a.2`, which is generated by $\sigma = (123\bar{1}\bar{2}\bar{3})$. We now show that this permutation representation doesn't lift to a divisor map with the properties listed in Proposition 3.10.

By Proposition 3.10(4), we have $\#\mathcal{P}_L \mid 3$ where $\mathcal{P}_L$ is the set of primes of $K = L$ dividing p. But then the action of G_A on $\mathcal{P}_L$ factors through C_3 and in particular the sequence

$$(\nu(\alpha_1), \nu(\sigma\alpha_1), ..., \nu(\sigma^5\alpha_1)) = \left(0, 0, \tfrac{1}{2}, 1, 1, \tfrac{1}{2}\right)$$

should be 3-periodic, a contradiction. □

Lemma 6.4. *An almost ordinary abelian threefold with Galois group D_6 has angle rank* 2.

Proof. From Table A.9, it suffices to show that the permutation representation `D6.6.t.a.1`, which has angle rank 3, does not occur. Suppose for the sake of contradiction that A is an almost ordinary abelian threefold with permutation representation `D6.6.t.a.1`. We exhibit a nontrivial multiplicative relation between the Frobenius eigenvalues (a contradiction to the angle rank being maximal) – in particular we show that $\alpha_1\overline{\alpha}_3/q$ is a root of unity.

Claim 1: $\#\mathcal{P}_L = 6$.
By Proposition 3.10(4), we have $\#\mathcal{P}_L \mid 6$. The order 6 cyclic subgroup of `D6.6.t.a.1` is generated by $\sigma = (1\bar{2}3\bar{1}2\bar{3})$. As in the proof of Lemma 6.4, because the sequence

$$(\nu(\alpha_1), \nu(\sigma\alpha_1), ..., \nu(\sigma^5\alpha_1)) = \left(0, 1, \tfrac{1}{2}, 1, 0, \tfrac{1}{2}\right)$$

is not periodic, we must have $\#\mathcal{P}_L = 6$.

Claim 2: *$\alpha_1\overline{\alpha}_3/q$ is a root of unity.*
Let $\mathfrak{p}_1$ be the prime of $\mathcal{O}_L$ corresponding to the valuation ν and let

$$(\mathfrak{p}_1, \overline{\mathfrak{p}}_2, \mathfrak{p}_3, \overline{\mathfrak{p}}_1, \mathfrak{p}_2, \overline{\mathfrak{p}}_3) = (\mathfrak{p}_1, \sigma(\mathfrak{p}_1), \ldots, \sigma^5(\mathfrak{p}_1)).$$

Because D_6 has a unique transitive permutation representation on a 6 element set, the action of D_6 on $\mathcal{P}_L$ is exactly the rigid symmetries of the hexagon as depicted in Figure 6.1. Because

$$(\nu(\alpha_1), \nu(\sigma\alpha_1), ..., \nu(\sigma^5\alpha_1)) = \left(0, 1, \tfrac{1}{2}, 1, 0, \tfrac{1}{2}\right),$$

we have

$$\mathrm{div}_A(\alpha_1) = en\left([\overline{\mathfrak{p}}_3] + \tfrac{1}{2}[\mathfrak{p}_2] + [\overline{\mathfrak{p}}_1] + \tfrac{1}{2}[\overline{\mathfrak{p}}_2]\right)$$

where $q = p^n$ and e is the ramification index of p in L. Now consider the element $\tau = (1\bar{3})(2\bar{2})(\bar{1}3) \in$ `D6.6.t.a.1` depicted in Figure 6.1. Note that τ is *not* complex conjugation and in fact $\tau(\alpha_1) = \overline{\alpha}_3$. The action of D_6 on $\mathcal{P}_L$ allows us to compute $\mathrm{div}_A(\overline{\alpha}_3)$, and it is:

$$\mathrm{div}_A(\overline{\alpha}_3) = \tau(\mathrm{div}_A(\alpha_1)) = en\left([\mathfrak{p}_1] + \tfrac{1}{2}[\overline{\mathfrak{q}}_2] + [\mathfrak{p}_3] + \tfrac{1}{2}[\mathfrak{p}_2]\right).$$

We have $(\overline{\alpha}_3\alpha_1)\mathcal{O}_L = q\mathcal{O}_L$, so $\overline{\alpha}_3\alpha_1 = qu$ for some unit $u \in \mathcal{O}_L^\times$. Because α_1 and $\overline{\alpha}_3$ are both q-Weil numbers, the unit $u = \alpha_1\overline{\alpha}_3/q$ has absolute value 1 over all complex places, thus it is a root of unity. □

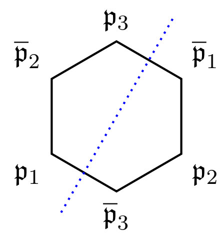

FIGURE 6.1. The action of G_A on $\mathcal{P}_L$. The permutation $\tau \in G_A$ is reflection with respect to the dotted line.

6.4. **Proof of Theorem 6.1 in the Newton polygon (C) case.** The following lemma completes the proof.

Lemma 6.5. *An abelian threefold with Newton polygon (C) cannot have Galois group $G_A \cong C_6$.*

Proof. Suppose for the sake of contradiction that A is an abelian threefold with Galois group $G_A \cong C_6$ and Newton polygon (C). It suffices to show that the image of the permutation representation is not conjugate to `C6.6.t.a.2`, which is generated by $\sigma = (123\bar{1}\bar{2}\bar{3})$.

By Proposition 3.10(4), we have $\#\mathcal{P}_L \mid 3$. But then the action of G_A on $\mathcal{P}_L$ factors through C_3 and in particular the sequence

$$(\nu(\alpha_1), \nu(\sigma\alpha_1), ..., \nu(\sigma^5\alpha_1)) = \left(0, \tfrac{1}{2}, \tfrac{1}{2}, 1, \tfrac{1}{2}, \tfrac{1}{2}\right)$$

should be 3-periodic, a contradiction. ☐

6.5. **Proof of Theorem 6.1 in the Newton polygon (D) case.** The following lemmas complete the proof.

Lemma 6.6. *A type (D) abelian threefold cannot have permutation representation whose image is w_A-conjugate to* `C6.6.t.a.2`.

Proof. Suppose for the sake of contradiction that A is a type (D) abelian threefold with permutation representation `C6.6.t.a.2`. By Proposition 3.10(4), we have $\#\mathcal{P}_L \mid 2$ where $\mathcal{P}_L$ is the set of primes of $K = L$ dividing p. But then the action of G_A on $\mathcal{P}_L$ factors through C_2 and in particular the sequence

$$(\nu(\alpha_1), \nu(\sigma\alpha_1), ..., \nu(\sigma^5\alpha_1)) = \left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{2}{3}, \tfrac{2}{3}\right)$$

should be 2-periodic, a contradiction. ☐

Lemma 6.7. *A type (D) abelian threefold cannot have permutation representation whose image is w_A-conjugate to* `D6.6.t.a.4`.

Proof. Suppose for the sake of contradiction that A is a type (D) abelian threefold with permutation representation `D6.6.t.a.4`. By Proposition 3.10(4), we have $\#\mathcal{P}_L \mid 4$. However `D6.6.t.a.4` contains the cyclic subgroup generated by $\sigma = (123\bar{1}\bar{2}\bar{3})$. But

$$(\nu(\alpha_1), \nu(\sigma\alpha_1), ..., \nu(\sigma^5\alpha_1)) = \left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{2}{3}, \tfrac{2}{3}\right)$$

is not periodic, so the action of C_6 on $\mathcal{P}_L$ is faithful and therefore $\#\mathcal{P}_L \geq 6$, a contradiction. ☐

6.6. **Proof of Theorem 6.1 in the supersingular case.**

Lemma 6.8. *The sextic field generated by the Frobenius eigenvalues of a simple supersingular abelian threefold must be* $\mathbf{Q}(\zeta_7)$ *or* $\mathbf{Q}(\zeta_9)$*, both of which have Galois group* C_6*.*

Proof. We follow the argument in [NR08, Proposition 2.1] (note that the characteristic of the base field in *loc. cit.* is 2). Supersingular abelian varieties have angle rank 0, so the sextic field K (as defined in Definition 2.7) is of the form $K \cong \mathbf{Q}(\zeta_m\sqrt{q})$ where ζ_m is a primitive m^{th}-root of unity. Choose m to be the smallest integer such that $\zeta_m\sqrt{q}$ generates K.

If q is a square, then the only cyclotomic fields of degree 6 are $\mathbf{Q}(\zeta_7)$ or $\mathbf{Q}(\zeta_9)$, so we are done.

Suppose now that q is not a square. If m is odd then K contains the field $\mathbf{Q}(\zeta_m)$. Thus $[\mathbf{Q}(\zeta_m) : \mathbf{Q}] \leq 6$ and $m = 3, 7, 9$. If $m = 3$ then $[K : \mathbf{Q}] \leq [\mathbf{Q}(\zeta_3, \sqrt{q}) : \mathbf{Q}] = 4$. If $m = 7, 9$, then $6 = [K : \mathbf{Q}] \geq [\mathbf{Q}(\zeta_m) : \mathbf{Q}] = 6$, so $K = \mathbf{Q}(\zeta_m)$.

Now suppose $m = 2n$ is even. Since K contains $\mathbf{Q}(\zeta_m^2) = \mathbf{Q}(\zeta_n)$ and in particular we have $n = 3, 4, 6, 7, 9, 14$. If $n = 7, 9, 14$ then $6 = [K : \mathbf{Q}] \geq [\mathbf{Q}(\zeta_n) : \mathbf{Q}] = 6$, so $K = \mathbf{Q}(\zeta_n)$. If $n = 3$, then $[K : \mathbf{Q}] \leq [\mathbf{Q}(\zeta_6, \sqrt{q}) : \mathbf{Q}] = 4$, contradiction. If $n = 6$, then $K = \mathbf{Q}(\zeta_{12}\sqrt{q})$. However, this is a quadratic extension of the quadratic field $\mathbf{Q}(\zeta_6)$, so $[K : \mathbf{Q}] = 4$, which is a contradiction. □

6.7. **When are simple abelian threefolds geometrically simple?** By the Honda–Tate theorem every supersingular abelian variety is not geometrically simple. Moreover, by Corollary 5.4 a simple ordinary threefold which is neither supersingular nor geometrically simple must be ordinary.

Suppose that A is a simple ordinary abelian threefold. In this case all possible permutation representations occur, and it suffices to show that if A is geometrically simple if it has angle rank 3 and not geometrically simple if it has angle rank 1.

Lemma 6.9. *Let* A *be a simple ordinary abelian threefold. Then the angle rank of* A *is* 3 *if* A *is geometrically simple and* 1 *otherwise.*

Proof. If A is geometrically simple, then $\delta_A = 3$ by [ABS24, Lemma 6.2.2]. If A is not geometrically simple then A has angle rank 1 by Corollary 5.4. □

7. Inverse Galois Problems

We state a slight refinement of a conjecture Dupuy, Kedlaya, Roe, and Vincent [DKRV21a, Conjecture 2.7].

Conjecture 7.1 (Refined inverse Galois problem for abelian varieties)**.** *Fix a prime number* p *and let* $G \subset W_{2d}$ *be a transitive subgroup containing the complex conjugation element. Then:*

(1) there exists an integer $r \geq 1$ *and a simple abelian variety* $A/\mathbf{F}_{p^r}$ *of dimension* d *such that* G *is* w_A*-conjugate to the image of the weighted permutation representation associated to* A*, and*

(2) the abelian variety A *may be taken to be ordinary.*

In particular, G *is isomorphic to the Galois group of some abelian variety* A*.*

We note that Conjecture 7.1 is a very strong statement. In particular it implies the inverse Galois problem holds even when we are restricted to totally real fields.

Proposition 7.2. *Assume Conjecture 7.1, then the inverse Galois problem holds for totally real fields. More precisely, let $d \geq 1$ and let $G \subset S_d$ be a transitive subgroup, then G is the Galois group of a polynomial $P^+(T)$ of degree d over $\mathbf{Q}$ and moreover the splitting field of $P^+(T)$ may be taken to be totally real.*

Proof. Let $G \subset S_d$ be a transitive subgroup, and consider the group $\widetilde{G} = C_2 \wr G$ equipped with its natural embedding $\widetilde{G} \hookrightarrow W_{2d}$. But $\widetilde{G}$ is a transitive subgroup of W_{2d} which contains the complex conjugation element, so by assumption it occurs as the Galois group of a q-Weil polynomial $P_A(T)$. Let $P_A^+(T)$ be the *trace polynomial* of $P_A(T)$ defined by the equation

$$P_A^+(T) = \prod_{\alpha} (T - (\alpha + \overline{\alpha})),$$

where α ranges over the roots of $P_A(T)$. It follows by construction that $\mathrm{Gal}(P_A^+(T))$ is isomorphic to G and that every root of $P_A^+(T)$ is real. □

Appendix A. Tables

In Table A.1 we record our labelling convention for subgroups of W_4 and W_6. In particular, we list every subgroup of W_4 and every transitive subgroup of W_6 containing complex conjugation.

A.1. **Abelian surfaces.** Tables A.2–A.7 record the possible subgroups $G \subset W_4$ which may occur as the (w_A-conjugacy class of the) images of weighted permutation representations associated to abelian surfaces A. For each case which does occur, we provide an example from the LMFDB [LMF24]. The tables are separated by Newton polygon (according to the conventions in Figure 1.1) and by whether A is simple.

A.1.1. *Simple abelian surfaces.* These cases are treated in Tables A.2–A.4. We do not list intransitive subgroups which do not occur as the image of the weighted permutation representation associated to a simple abelian surface – note that an intransitive subgroup can only occur for the supersingular abelian surface in Lemma 3.5(2). In each case we record whether every isogeny class of abelian surfaces with the recorded image of the weighted permutation representation is geometrically simple.

A.1.2. *Non-simple abelian surfaces.* These cases are treated in Tables A.5–A.7. Since transitive subgroups of W_4 cannot occur as the image of the weighted permutation representation associated to an abelian surface, we do not record them.

A.2. **Abelian threefolds.** Tables A.8–A.12 record the possible subgroups $G \subset W_6$ which may occur as the (w_A-conjugacy class of the) images of weighted permutation representations associated to simple abelian threefolds A. For each case which does occur, we provide an example from the LMFDB [LMF24]. The tables are separated by Newton polygon (according to the conventions in Figure 1.3).

Label of G	Generators of G	Label of G	Generators of G
W4.4.t.a.1	W_4	W6.6.t.a.1	W_6
V4.4.t.a.1	$\iota, (12)(\bar{1}\bar{2})$	6T6.6.t.a.1	$(123)(\bar{1}\bar{2}\bar{3})$, $(1\bar{1})$, $(2\bar{2})$, $(3\bar{3})$
C4.4.t.a.1	$(12\bar{1}\bar{2})$	D6.6.t.a.1	$(1\bar{2}3\bar{1}2\bar{3}), (23)(\bar{2}\bar{3})$
V4.4.nt.a.1	$(1\bar{1}), (2\bar{2})$	D6.6.t.a.2	$(12\bar{3}\bar{1}\bar{2}3)$, $(23)(\bar{2}\bar{3})$
C2.4.nt.a.1	ι	D6.6.t.a.3	$(1\bar{2}\bar{3}\bar{1}23)$, $(2\bar{3})(\bar{2}3)$
C2.4.nt.b.1	$(1\bar{1})$	D6.6.t.a.4	$(123\bar{1}\bar{2}\bar{3})$, $(2\bar{3})(\bar{2}3)$
C2.4.nt.b.2	$(2\bar{2})$	C6.6.t.a.1	$(1\bar{2}3\bar{1}2\bar{3})$
C2.4.nt.c.1	$(12)(\bar{1}\bar{2})$	C6.6.t.a.2	$(123\bar{1}\bar{2}\bar{3})$
C2.4.nt.c.2	$(1\bar{2})(\bar{1}2)$	C6.6.t.a.3	$(1\bar{3}\bar{1}\bar{2}3)$
C1.4.nt.a.1	id	C6.6.t.a.4	$(1\bar{2}\bar{3}\bar{1}23)$

TABLE A.1. Labels for subgroups of W_4 (left) and subgroups of W_6 (right).

w_A-conjugacy class	Angle rank	Occurs	Geometrically simple	Example(s)
W4.4.t.a.1	2	Yes	Yes	2.2.ac_d
V4.4.t.a.1	1	Yes	No	2.2.ad_f
C4.4.t.a.1	2	Yes	Yes	2.3.ad_f

TABLE A.2. The images of the weighted permutation representations associated to a simple ordinary abelian surface (Newton polygon (A) in Figure 1.1).

w_A-conjugacy class	Angle rank	Occurs	Geometrically simple	Example(s)
W4.4.t.a.1	2	Yes	Yes	2.2.ab_a
V4.4.t.a.1	2	No		
C4.4.t.a.1	2	No		

TABLE A.3. The images of the weighted permutation representations associated to simple almost ordinary abelian surfaces (Newton polygon (B) in Figure 1.1).

w_A-conjugacy class	Angle rank	Occurs	Geometrically simple	Example(s)
W4.4.t.a.1	2	No		
V4.4.t.a.1	0	Yes	No	2.2.ac_c
C4.4.t.a.1	0	Yes	No	2.4.ac_e
C2.4.nt.c.1 C2.4.nt.c.2	0	Yes	No	2.2.a_ae

TABLE A.4. The images of the weighted permutation representations associated to simple supersingular abelian surfaces (Newton polygon (C) in Figure 1.1).

w_A-conjugacy class	Angle rank	Occurs	Example(s)
`V4.4.nt.a.1`	1	Yes	`2.3.ad_i`
`C2.4.nt.a.1`	1	Yes	`2.2.a_d`
`C2.4.nt.b.1` `C2.4.nt.b.2`	1	No	
`C2.4.nt.c.1`	1	No	
`C2.4.nt.c.2`	1	No	
`C1.4.nt.a.1`	0	No	

TABLE A.5. The images of the weighted permutation representations associated to non-simple ordinary abelian surfaces (Newton polygon (A) in Figure 1.2).

w_A-conjugacy class	Angle rank	Occurs	Example(s)
`V4.4.nt.a.1`	1	Yes	`2.2.ad_g`
`C2.4.nt.a.1`	1	No	
`C2.4.nt.b.1`	1	No	
`C2.4.nt.b.2`	1	Yes	`2.4.ah_u`
`C2.4.nt.c.1`	1	No	
`C2.4.nt.c.2`	1	No	
`C1.4.nt.a.1`	0	No	

TABLE A.6. The images of the weighted permutation representations associated to a non-simple almost ordinary abelian surfaces (Newton polygon (B) in Figure 1.2).

w_A-conjugacy class	Angle rank	Occurs	Example(s)
`V4.4.nt.a.1`	0	Yes	`2.2.ac_e`
`C2.4.nt.a.1`	0	Yes	`2.2.a_a`
`C2.4.nt.b.1` `C2.4.nt.b.2`	0	Yes	`2.4.ag_q`
`C2.4.nt.c.1` `C2.4.nt.c.2`	0	No	
`C1.4.nt.a.1`	0	Yes	`2.4.a_ai`

TABLE A.7. The images of the weighted permutation representations associated to non-simple supersingular abelian surfaces (Newton polygon (C) in Figure 1.2).

w_A-conjugacy class	Angle rank	Occurs	Geometrically simple	Example(s)
W6.6.t.a.1	3	Yes	Yes	3.2.ad_f_ah
6T6.6.t.a.1	3	Yes	Yes	3.2.ad_g_aj
D6.6.t.a.1	1	Yes	No	3.2.a_a_ad
D6.6.t.a.2 D6.6.t.a.3 D6.6.t.a.4	3	Yes	Yes	3.2.ac_a_d
C6.6.t.a.1	1	Yes	No	3.2.ae_j_ap
C6.6.t.a.2 C6.6.t.a.3 C6.6.t.a.4	3	Yes	Yes	3.7.ak_bw_afv

TABLE A.8. The images of the weighted permutation representations associated to simple ordinary abelian threefolds (Newton polygon (A) in Figure 1.3).

w_A-conjugacy class	Angle rank	Occurs	Geometrically simple	Example
W6.6.t.a.1	3	Yes	Yes	3.2.ab_ab_c
6T6.6.t.a.1	3	Yes	Yes	3.4.ac_ab_g
D6.6.t.a.1 D6.6.t.a.3	3	No		
D6.6.t.a.2 D6.6.t.a.4	2	Yes	Yes	3.2.ac_b_a
C6.6.t.a.1 C6.6.t.a.4	3	No		
C6.6.t.a.2 C6.6.t.a.3	2	No		

TABLE A.9. The images of the weighted permutation representations associated to simple almost ordinary abelian threefold (Newton polygon (B) in Figure 1.3).

<table>
<tr><th>w-conjugacy class</th><th>Angle rank</th><th>Occurs</th><th>Geometrically simple</th><th>Example</th></tr>
<tr><td>W6.6.t.a.1</td><td>3</td><td>Yes</td><td>Yes</td><td>3.2.ab_a_a</td></tr>
<tr><td>6T6.6.t.a.1</td><td>3</td><td>Yes</td><td>Yes</td><td>3.4.ab_c_a</td></tr>
<tr><td>D6.6.t.a.1
D6.6.t.a.2
D6.6.t.a.3
D6.6.t.a.4</td><td>3</td><td>Yes</td><td>Yes</td><td>3.4.ab_a_ae</td></tr>
<tr><td>C6.6.t.a.1
C6.6.t.a.2
C6.6.t.a.3
C6.6.t.a.4</td><td>3</td><td>No</td><td></td><td></td></tr>
</table>

TABLE A.10. The images of the weighted permutation representations associated to a simple abelian threefolds with Newton polygon (C) in Figure 1.3.

<table>
<tr><th>w-conjugacy class</th><th>Angle rank</th><th>Occurs</th><th>Geometrically simple</th><th>Example</th></tr>
<tr><td>W6.6.t.a.1</td><td>3</td><td>Yes</td><td>Yes</td><td>3.2.ac_c_ac</td></tr>
<tr><td>6T6.6.t.a.1</td><td>3</td><td>Yes</td><td>Yes</td><td>3.3.ad_j_ap</td></tr>
<tr><td>D6.6.t.a.1</td><td>1</td><td>Yes</td><td>Yes</td><td>3.2.a_a_ac</td></tr>
<tr><td>D6.6.t.a.2
D6.6.t.a.3
D6.6.t.a.4</td><td>3</td><td>No</td><td></td><td></td></tr>
<tr><td>C6.6.t.a.1</td><td>1</td><td>Yes</td><td>Yes</td><td>3.7.a_a_abj</td></tr>
<tr><td>C6.6.t.a.2
C6.6.t.a.3
C6.6.t.a.4</td><td>3</td><td>No</td><td></td><td></td></tr>
</table>

TABLE A.11. The images of the weighted permutation representations associated to simple abelian threefolds with Newton polygon (D) in Figure 1.3.

w-conjugacy class	Angle rank	Occurs	Geometrically simple	Example
W6.6.t.a.1	0	No		
6T6.6.t.a.1	0	No		
D6.6.t.a.1 D6.6.t.a.2 D6.6.t.a.3 D6.6.t.a.4	0	No		
C6.6.t.a.1 C6.6.t.a.2 C6.6.t.a.3 C6.6.t.a.4	0	Yes	No	3.3.a_a_aj

TABLE A.12. The images of the weighted permutation representations associated to simple almost ordinary abelian threefolds (Newton polygon (E) in Figure 1.3).

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